Inter (Part-II) 2019

Mathematics	Group-II	PAPER: II
Time: 2.30 Hours	(SUBJECTIVE TYPE)	Marks: 80

SECTION-I

2. Write short answers to any EIGHT (8) questions: (16)

(i) Define implicit function.

Ans If x and y are so mixed up and y cannot expressed in terms of the independent variable x, then y is called an implicit function. For example,

1. $x^2 + xy + y^2 = 0$

2. $\dfrac{xy^2 - y + 9}{xy} = +1$ are implicit functions of x and y.

Symbolically, it is written as $f(x, y) = 0$.

(ii) $f(x) = 3x^4 - 2x^2$, $g(x) = \dfrac{2}{\sqrt{x}}$, find $f(g(x))$.

Ans $f(x) = 3x^4 - 2x^2 \qquad g(x) = \dfrac{2}{\sqrt{x}}$

$$fog\,(x) = f[g(x)] = 3\left(\frac{2}{\sqrt{x}}\right)^4 - 2\left(\frac{4}{\sqrt{x}}\right)^2$$

$$= 3\left(\frac{16}{x^2}\right) - 2\left(\frac{4}{x}\right) = \frac{48}{x^2} - \frac{8}{x}$$

$$= \frac{48 - 8x}{x^2} = \frac{8(6 - x)}{x^2}$$

(iii) Evaluate $\lim\limits_{x \to 2} \dfrac{\sqrt{x} - \sqrt{2}}{x - 2}$.

Ans

$$= \lim_{x \to 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2} \times \frac{\sqrt{x} + \sqrt{2}}{\sqrt{x} + \sqrt{2}}$$

$$= \lim_{x \to 2} \frac{x - 2}{(x - 2)(\sqrt{x} + \sqrt{2})}$$

$$= \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}$$

(iv) Find derivative by definition of x^2.

Ans $f(x) = x^2$

1. $f(x+\delta x) = (x+\delta x)^2$

2. $f(x+\delta x) - f(x) = (x+\delta x)^2 - x^2$

$$= x^2 + 2x\,\delta x + (\delta x)^2 - x^2$$

$$= 2x\,\delta x + (\delta x)^2 = (2x + \delta x)\,\delta x$$

3. $$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{(2x+\delta x)\,\delta x}{\delta x} = 2x + \delta x \qquad (\delta x \neq 0)$$

4. $$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\lim_{\delta x \to 0} (2x + \delta x) = 2x$$

i.e., $f'(x) = 2x$

(v) Differentiate w.r.t. 'x' $\sqrt{\dfrac{a-x}{a+x}}$.

Ans Let $y = \sqrt{\dfrac{a-x}{a+x}}$ and $u = \dfrac{a-x}{a+x}$

Then $y = u^{1/2}$

Now, $$\frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1} = \frac{1}{2}u^{-\frac{1}{2}}$$

and $$\frac{du}{dx} = \frac{d}{dx}\left[\frac{a-x}{a+x}\right]$$

$$= \frac{\left[\frac{d}{dx}(a-x)\right](a+x) - (a-x)\left[\frac{d}{dx}(a+x)\right]}{(a+x)^2}$$

$$= \frac{(0-1)(a+x) - (a-x)(0+1)}{(a+x)^2} = \frac{-a-x-a+x}{(a+x)^2}$$

$$= \frac{-2a}{(a+x)^2}$$

Using the formula,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}, \quad \text{we have}$$

$$\frac{d}{dx}\left(\sqrt{\frac{a-x}{a+x}}\right) = \frac{1}{2}u^{-\frac{1}{2}}\left[\frac{-2a}{(a+x)^2}\right].$$

$$= \frac{1}{2}\left(\frac{a-x}{a+x}\right)^{1/2} \times \frac{-2a}{(a+x)^2} \qquad \left(\because\ u = \frac{a-x}{a+x}\right)$$

$$= \frac{(a-x)^{-1/2}}{(a+x)^{-1/2}} \times \frac{-a}{(a+x)^2} = \frac{-a}{(a-x)^{1/2}(a+x)^{3/2}}$$

(vi) **Find $\frac{dy}{dx}$, if $x^2 - 4xy - 5y = 0$.**

Ans $x^2 - 4xy - 5y = 0$

Differentiating the equation w.r.t x:

$$\frac{d}{dx}(x^2 - 4xy - 5y) = 0$$

$$\frac{d}{dx}x^2 - \frac{d}{dx}(4xy) - \frac{d}{dx}(5y) = 0$$

$$2x - 4\left[x\frac{d}{dx}(y) + y\frac{d}{dx}(x)\right] - 5\frac{d}{dx}(y) = 0$$

$$2x - 4\left(x\frac{dy}{dx} + y\right) - 5\frac{dy}{dx} = 0$$

$$2x - 4x\frac{dy}{dx} - 4y - 5\frac{dy}{dx} = 0$$

$$-\frac{dy}{dx}(4x+5) = 4y - 2x$$

$$\frac{dy}{dx} = -\frac{(4y-2x)}{4x+5}$$

$$= \frac{2x-4y}{4x+5} = \frac{2(x-2y)}{4x+5}$$

(vii) **Prove that $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$.**

Ans Let $y = \cot^{-1} x \Rightarrow y = \frac{1}{\cot}x$

$\therefore \quad \cot y = x$ (i)

Differentiating w.r.t. x,

$$\frac{d}{dx}(\cot y) = \frac{d}{dx}(x)$$

$$-\text{cosec}^2 y \cdot \frac{dy}{dx} = 1$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{\text{cosec}^2 y}$$

$$= \frac{1}{1+\cot^2 y} = -\frac{1}{1+x^2}$$ **Proved.**

(viii) Find $\frac{dy}{dx}$, if $y = x\cos y$.

Ans

$$y = x\cos y$$

Differentiate w.r.t. x.

$$\frac{dy}{dx} = \frac{d}{dx}(x\cos y)$$

$$= x\frac{d}{dx}(\cos y) + \cos y\frac{d}{dx}(x)$$

$$= x\left(-\sin y \cdot \frac{dy}{dx}\right) + \cos y\,(1)$$

$$= -x\sin y\frac{dy}{dx} + \cos y$$

$$= \frac{dy}{dx} + x\sin y\frac{dy}{dx} = \cos y = \frac{dy}{dx}(1 + x\sin y) = \cos y$$

$$\frac{dy}{dx} = \frac{\cos y}{1 + x\sin y}$$

(ix) Find $f'(x)$, if $f(x) = \sqrt{\ln(e^{2x} + e^{-2x})}$.

Ans Let $u = e^{2x} + e^{-2x}$

Then $f(x) = (\ln u)^{1/2}$ and

$$f'(x) = \frac{d}{dx}(\ln u)^{1/2} = \frac{d}{du}(\ln u)^{1/2} \times \frac{du}{dx} = \left[\frac{1}{2}(\ln u)^{1/2-1}\frac{d}{du}(\ln u)\right] \times \frac{d}{dx}(e^{2x} + e^{-2x})$$

$$= \left(\frac{1}{2}\cdot\frac{1}{(\ln u)^{1/2}}\cdot\frac{1}{u}\right)\cdot(e^{2x}\cdot 2 + e^{-2x}\cdot(2))$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{\ln(e^{2x} + e^{-2x})}}\cdot\frac{1}{e^{2x} + e^{-2x}}\cdot 2(e^{2x} - e^{-2x})$$

$$= \frac{e^{2x} + e^{-2x}}{\sqrt{\ln(e^{2x} + e^{-2x})} \times (e^{2x} + e^{-2x})}$$

(x) Find y_2, if $x = at^2$, $y = bt^4$.

Ans

$$x = at^2 \quad ; \quad y = bt^4$$

$$\frac{dx}{dt} = \frac{d}{dt}(at^2) \quad ; \quad \frac{dy}{dt} = \frac{d}{dt}(bt^4)$$

$$= 2at \quad ; \quad = 4bt^3$$

By Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 4bt^3 \times \frac{1}{2at}$$

$$\therefore \quad y_1 = \frac{2bt^2}{a}$$

$$\frac{d^2y}{dx^2} = \frac{2b}{a} \cdot \frac{d}{dx}(t^2)$$

$$\therefore \quad y_2 = \frac{2b}{a} \cdot (2t) \cdot \frac{d}{dx}(t)$$

$$= \frac{4bt}{a} \cdot \frac{dt}{dx}$$

$$= \frac{4bt}{a} \times \frac{1}{2at} = \frac{2b}{a^2}$$

(xi) Define Maclaurin series.

Ans We have $a_0 = f(0)$, $a_1 = f'(0)$, $a_2 = f''\frac{(0)}{2!}$

$$a_3 = \frac{f'''(0)}{3!}, \; a_4 = \frac{f''''(0)}{4!}$$

Following the above pattern, we can write

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Thus substituting these values in the power series, we have

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^n(0)}{n!}x^n + \ldots$$

This expansion of f(x) is called the Maclaurin series.

(xii) Determine the interval in which f(x) is increasing or decreasing if f(x) = sin x, x ∈ (0, π).

Ans

$$f(x) = \sin x$$
$$f'(x) = \cos x$$

$f'(x)$ is +ve $\forall\, x \in (0, \frac{\pi}{2})$

$\therefore$ f(x) is increasing.

$f'(x) = \cos x$ is –ve for all $x \in (\frac{\pi}{2}, \pi)$

then t is decreasing.

3. Write short answers to any EIGHT (8) questions: (16)

(i) Using differential, find $\frac{dy}{dx}$ when $xy - \ln x = c$.

Ans

$$xy - \ln x = c$$

Taking differential on both sides,

$$d(xy - \ln x) = d(c)$$

$$x\,dy + y\,dx - \frac{1}{x}dx = 0$$

$$x\,dy + y\,dx - \frac{1}{x}dx = 0$$

$$x\,dy = \frac{dx}{x} - y\,dx$$

$$= dx\left(\frac{1}{x} - y\right)$$

$$\frac{dy}{dx} = \frac{\frac{1 - xy}{x}}{x} = \frac{1 - xy}{x^2}$$

(ii) Evaluate $\int \frac{(\sin x + \cos^2 x)}{\cos^2 x \,.\, \sin x} dx.$

Ans

$$= \int \left(\frac{\sin x}{\cos^2 x \,.\, \sin x} + \frac{\cos^2 x}{\cos^2 x \,.\, \sin x}\right) dx$$

$$= \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin x}\right) dx$$

$$= \int \sec^2 x\,dx + \int \operatorname{cosec} x\,dx$$

$$= \tan x + \ln|\operatorname{cosec} x - \cot x| + c$$

(iii) Find $\int x(\sqrt{x} + 1)\,dx\ ;\ x > 0.$

Ans

$$\int x \,.\, x^{1/2} dx + \int x\,dx$$

$$\int x^{3/2} dx + \frac{x^2}{2} + c$$

$$= \frac{x^{5/2}}{\frac{5}{2}} + \frac{x^2}{2} + c$$

$$= \frac{2x^{5/2}}{5} + \frac{x^2}{2} + c$$

(iv) Evaluate $\int a^{x^2} . x\,dx\ ;\ a > 1.$

Ans Put $x^2 = t$, then $x\,dx = \frac{1}{2}dt$

Thus $\int a^{x^2} . x\,dx = \int a^t \times \frac{1}{2}dt$

$$= \frac{1}{2}\int a^t\, dt = \frac{1}{2}\frac{a^t}{\ln a} + c = \frac{a^{x^2}}{2\ln a} + c$$

(v) Find the anti-derivative of x . e^x.

Ans $\int x . e^x . dx$ Let $u = x$ and $dv = e^x\, dx$

Then $du = 1 . dx$ and $v = e^x$

Applying the formula for integration by parts, we have

$$\int x\, e^x\, dx = x\, e^x - \int e^x \times 1\, dx$$

$$= x\, e^x - e^x + c$$

(vi) Evaluate $\int e^x (\cos x + \sin x)\, dx$.

Ans $= \int e^x \cos x\, dx + \int e^x \sin x\, dx$

Integrating 1st integral by parts,

$$= e^x \int \cos x\, dx - \int \left[\frac{d}{dx}(e^x) . \int \cos x\, dx\right] dx + \int e^x \sin x\, dx + c$$

$$= e^x . \sin x - \int e^x . \sin x\, dx + \int e^x \sin x\, dx + c$$

$$= e^x \sin x + c$$

(vii) State 'Fundamental Theorem' of calculus.

Ans If f is continuous on [a , b] and $\phi'(x) = f(x)$, that is, $\phi(x)$ is any anti-derivative of f on [a, b], then

$$\int_a^b f(x)\, dx = \phi(b) - \phi(a)$$

Note that the difference $\phi(b) - \phi(a)$ is independent of the choice of anti-derivative of the function f.

(viii) Compute $\int_{-1}^{1} (x^{1/3} + 1)\, dx$.

Ans $\int_{-1}^{1} (x^{1/3} + 1)\, dx = \int_{-1}^{1} x^{1/3}\, dx + \int_{-1}^{1} dx$

$$= \left[\frac{x^{4/3}}{\frac{4}{3}}\right]_{-1}^{1} + [x]_{-1}^{1}$$

$$= \frac{3}{4}\left[(1)^{4/3} - (-1)^{4/3}\right] + [(1) - (-1)]$$

$$= \frac{3}{4}[1 - 1] + 2 = 2$$

(ix) Find the area above x-axis and under the curve $y = 5 - x^2$ from $x = -1$ to $x = 2$.

Ans Area $= \int_{-1}^{2} (5 - x^2)\, dx$

$$= 5\int_{-1}^{2} dx - \int_{-1}^{2} x^2\, dx$$

$$= 5[x]_{-1}^{2} - \left[\frac{x^3}{3}\right]_{-1}^{2}$$

$$= 5[2 - (-1)] - \frac{1}{3}[8 - (-1)^3]$$

$$= 5[3] - \frac{1}{3}[8 + 1]$$

$$= 15 - \frac{1}{3}(9) = 15 - 3 = 12 \text{ sq. units}$$

(x) Solve the differential equation $\sin y \,.\, \operatorname{cosec} x \,.\, \frac{dy}{dx} = 1$.

Ans

$$\sin y \operatorname{cosec} x \frac{dy}{dx} = 1$$

$$\sin y\, dy = \frac{dx}{\operatorname{cosec} x}$$

$$\sin y\, dy = \sin x\, dx$$

$$\int \sin y\, dy = \int \sin x\, dx + c'$$

$$-\cos y = -\cos x + c'$$

$$\cos y = \cos x - c'$$

$$\cos y = \cos x + c$$

(xi) Define 'decision variables'.

Ans The variables used in the system of linear inequalities relating to the problems of everyday life are non-negative constraints. These non-negative constraints play an important role for taking decision. So these variables are decision variables.

(xii) Graph solution set of inequality $2x + y \geq 2$ in $x - y$ plane.

Ans We draw the graph of the line $2x + y = 2$ joining the points (1, 0) and (0, 2). The point (0, 0) does not satisfy the inequality $2x + y > 0$ because $2(0) + 0 = 0 \not> 2$. Thus the graph of the inequality $2x + y \geq 2$ is the closed half not on the origin-side of the line $2x + y = 2$.

The intersection point (2, –2) can be found by solving the equation $2x + y = 2$.

4. Write short answers to any NINE (9) questions: (18)

(i) Find the coordinates of the point that divides the join of A(–6, 3) and B(5, –2) internally in ratio 2 : 3.

Ans Here $k_1 = 2, k_2 = 3, x_1 = -6, x_2 = 5$

By the formula, we have $y_1 = 3, y_2 = -2$

$$x = \frac{(2 \times 5) + 3(-6)}{2+3} = \frac{10-18}{5} = \frac{-8}{5}$$

$$\text{and} \quad y = \frac{(2(-2)) + 3 \times 3}{2+3} = \frac{-4+9}{5} = \frac{5}{5} = 1$$

Coordinates of the required points are $\left(\frac{-8}{5}, 1\right)$.

(ii) Find the slope and inclination of the line joining the points A(–2, 4) and B(5, 11).

Ans Let A(–2, 4) and B(5, 11)

$$\text{Slope of AB} = m = \frac{11-4}{5-(-2)} = \frac{7}{5+2} = \frac{7}{7} = 1$$

$$m = \tan\theta = 1 \Rightarrow \text{inclination} = \theta = \tan^{-1}(1) = 45^\circ$$

(iii) By means of slopes show that points A(–1, –3), B(1, 5) and C(2, 9) are collinear.

Ans We know that the points A, B and C are collinear, if the line AB and BC have the same slopes.

$$\text{Slope of AB} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-(-3)}{1-(-1)} = \frac{5+3}{1+2} = \frac{8}{2} = 4$$

$$\text{and} \quad \text{Slope of BC} = \frac{9-5}{2-1} = \frac{4}{1} = 4$$

$\therefore$ Slope of AB = Slope of BC

Thus A, B and C are collinear.

(iv) Find equation of the line through (–4, 7) and parallel to the line 2x – 7y + 4 = 0.

Ans

$$2x - 7y + 4 = 0$$
$$-7y = -2x - 4$$
$$7y = 2x + 4$$
$$\frac{7y}{7} = \frac{2x}{7} + \frac{4}{7}$$

$$y = \frac{2}{7}x + \frac{4}{7}$$

$\therefore$ Slope of required linear = $m = \frac{2}{7}$

Equation of line through (–4, 7) with $m = \frac{2}{7}$ is

$$y - y_1 = m(x - x_1)$$

$$y - 7 = \frac{2}{7}(x - (-4))$$

$$y - 7 = \frac{2}{7}(x + 4)$$

$$7y - 49 = 2x + 8$$

$$2x - 7y = 49 + 8 \Rightarrow \quad 2x - 7y = 57$$

$$\Rightarrow 2x - 7y - 57 = 0$$

(v) Find equation of circle with centre at (5, –2) and radius 4.

Ans Here $c(h, k) = c(5, -2)$ and $r = 4$

Equation of required circle is

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - 5)^2 + (y - (-2)^2) = (4)^2$$

$$(x - 5)^2 + (y + 2)^2 = (4)^2$$

$$x^2 + 25 - 10x + y^2 + 4y + 4 = 16$$

$$x^2 + y^2 - 10x + 4y + 29 = 16$$

$$x^2 + y^2 - 10x + 4y + 29 - 16 = 0$$

$$x^2 + y^2 - 10x + 4y + 13 = 0$$

(vi) Find focus and vertex of the parabola $y^2 = -8(x - 3)$.

Ans $y^2 = -8(x - 3) \Rightarrow y^2 = -8x$ where $x = x - 3$

Comparing it with $y^2 = -4ax$

$$-4a = -8$$

$$4a = 8$$

$$a = \frac{8}{4} = 2$$

Coordinates of focus: $(-a, 0)$

$$x = -a, \ y = 0$$

$$x - 3 = -2$$

$$x = 3 - 2 = 1 \qquad f(1, 0)$$

Coordinates of vertex, (0, 0)

$$x = 0, \ y = 0$$

$$x - 3 = 0 \qquad v(0, 0)$$

(vii) Find equation of tangent to the parabola $x^2 = 16y$ at the point whose abscissa is 8.

Ans Since $x = 8$ lies on the parabola,

Substituting this value of x into the given equation, we find

$$x^2 = 16y$$
$$(8)^2 = 16y$$
$$64 = 16y \Rightarrow y = 4$$

Thus we have to find equations of tangent and normal at (8, 4).

Slope of the tangent to the parabola at (8, 4) is 1. An equation of the tangent the parabola at (8, 4) is

$$y - 4 = x - 8$$
$$x - y + 4 - 8 = 0$$
$$x - y - 4 = 0$$

Slope of the normal at (8, 4) is −1. Therefore, equation of the normal at the given point is

$$y - 4 = -(x - 8)$$
$$y - 4 = +8 - x$$
$$x + y - 4 - 8 = 0$$
$$x + y - 12 = 0$$

(viii) Find foci and vertices of the ellipse $25x^2 + 9y^2 = 225$.

Ans
$$25x^2 + 9y^2 = 225$$

Dividing both sides by 225

$$\frac{25x^2}{225} + \frac{9y^2}{225} = \frac{225}{225}$$
$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

Here, $a^2 = 25 \Rightarrow a = \pm 5$

$b^2 = 9 \Rightarrow b = \pm 3$

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{25 - 9}{25} \Rightarrow \frac{16}{25} \Rightarrow e = \frac{4}{5}$$

$$ae = 5\left(\frac{4}{5}\right) = 4$$

$$\frac{a}{e} = \frac{5}{\frac{4}{5}} = \frac{5 \times 5}{4} = \frac{25}{4}$$

$f = (0, \pm ae) = (0, \pm 4)$

$V(0, \pm a) = V(0 \pm 5)$

(ix) Find the angle between the vectors $\underline{u} = 2\underline{i} - \underline{j} + \underline{k}$ and $\underline{v} = -\underline{i} + \underline{j}$.

Ans

$$\underline{u} = 2\underline{i} - \underline{j} + \underline{k}$$

$$\underline{v} = -\underline{i} + \underline{j}$$

$$\underline{u}.\underline{v} = (2\underline{i} - \underline{j} + \underline{k}) . (-\underline{i} + \underline{j} + 0\underline{k})$$

$$= (2)(-1) + (-1)(1) + (1)(0) = -3$$

$$\therefore \quad |\underline{u}| = |2\underline{i} - \underline{j} + \underline{k}| = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{6}$$

$$\text{and} \quad |\underline{v}| = |-\underline{i} + \underline{j} + 0k| = \sqrt{(-1)^2 + (1)^2 + (0)^2} = \sqrt{2}$$

$$\cos\theta = \frac{\underline{u}\ .\ \underline{v}}{|\underline{u}|\ .\ |\underline{v}|}$$

$$\cos\theta = \frac{-3}{\sqrt{6}\sqrt{2}} - \frac{\sqrt{3}}{2}$$

$$\therefore \quad \theta = \frac{5\pi}{6}$$

(x) Find scalar α so that the vectors $2\underline{i} + \alpha\,\underline{j} + 5\underline{k}$ and $3\underline{i} + \underline{j} + \alpha\,\underline{k}$ are perpendicular.

Ans Let $\quad \underline{u} = 2\underline{i} + \alpha\underline{j} + 5\underline{k}$

and $\quad \underline{v} = 3\underline{i} + \underline{j} + \alpha\underline{k}$

It is given that $\underline{u}$ and $\underline{v}$ are perpendicular.

$$\therefore \quad \underline{u}\ .\ \underline{v} = 0$$

$$\Rightarrow \quad (2\underline{i} + \alpha\underline{j} + 5\underline{k}) . (3\underline{i} + \underline{j} + \alpha\,\underline{k}) = 0$$

$$\Rightarrow \quad 6 + \alpha + 5\alpha = 0$$

$$6\alpha = -6$$

$$\alpha = \frac{-6}{6} = -1$$

$$\therefore \quad \alpha = -1$$

(xi) If $\underline{v}$ is a vector for which $\underline{v}\ .\ \underline{i} = 0$, $\underline{v}\ .\ \underline{j} = 0$, $\underline{v}\ .\ \underline{k} = 0$, find $\underline{v}$.

Ans Let $\quad \underline{v} = a\underline{i} + b\underline{j} + c\underline{k} \qquad$ (i)

$$\because \quad \underline{v}\ .\ \underline{i} = 0$$

$$\therefore \quad (a\underline{i} + b\underline{j} + c\underline{k})(\underline{i} - 0\underline{j} + 0\underline{k}) = 0$$

$$(a)(1) + (b)(0) + (c)(0) = 0$$

$$a = 0 \qquad \text{(ii)}$$

$$\because \quad \underline{v}\ .\ \underline{j} = 0$$

$$\therefore \quad (a\underline{i} + b\underline{j} + c\underline{k}) . (0\underline{i} + \underline{j} + 0\underline{k}) = 0$$

$\therefore \quad (a)(0) + b(1) + c(0) = 0$

$\therefore \quad b = 0 \qquad \text{(iii)}$

$\because \quad \underline{v} \cdot \underline{k} = 0$

$\therefore \quad (a\underline{i} + b\underline{j} + c\underline{k}) \cdot (0\underline{i} + 0\underline{j} + \underline{k}) = 0$

$\therefore \quad (a \times 0) + (b \times 0) + (c \times 1) = 0$

$\therefore \quad c = 0 \qquad \text{(iv)}$

Putting (ii) (iii) and (iv) in eq. (i),

$v = (0)\underline{i} + 0(\underline{j}) + 0(\underline{k})$

$\underline{v} = 0$ (Zero / Null vector)

(xii) Prove that $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b}) = 0$.

Ans $\underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} + \underline{b} \times \underline{a} + \underline{c} \times \underline{a} + \underline{c} \times \underline{b} = 0$

$\underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} - \underline{a} \times \underline{b} - \underline{a} \times \underline{c} - \underline{b} \times \underline{c} = 0$

$= 0$ R.H.S

(xiii) Find the value of α, so that $\alpha\underline{i} + \underline{j}$, $\underline{i} + \underline{j} + 3\underline{k}$ and $2\underline{i} + \underline{j} - 2\underline{k}$ are coplanar.

Ans Let $\underline{u} = a\underline{i} + \underline{j}$ $\quad \underline{v} = \underline{i} + \underline{j} + 3\underline{k}$ and

$\underline{w} = 2\underline{i} + \underline{j} - 2\underline{k}$

Triple scalar product

$$[\underline{u}\ \underline{v}\ \underline{w}] = \begin{bmatrix} \alpha & 1 & 0 \\ 1 & 1 & 3 \\ 2 & 1 & -2 \end{bmatrix}$$

$= \alpha(-2-3) + -1(-2-6) + 0(1-2)$

$= -5\alpha + 8$

The vectors will be coplaner if

$-5\alpha + 8 = 0$

$-5\alpha = -8$

$\alpha = \dfrac{8}{5}$

SECTION-II

NOTE: Attempt any Three (3) questions.

Q.5.(a) If $f(x) = \begin{cases} 3x & \text{if } x \le -2 \\ x^2 - 1 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \ge 2 \end{cases}$ **(5)**

discuss continuity at $x = 2$ and $x = -2$.

Ans (i) At $x = 2$

(a) $f(2) = 3$ (Defined)

(b) $\text{L.H.L.} = \lim_{x\to 2^-} (x^2 - 1) = (2)^2 - 1 = 3$

$\text{R.H.L.} = \lim_{x\to 2^+} (3) = 3$

$\therefore$ L.H.L. = R.H.L.

(c) $\lim_{x\to 2} f(x) = 3$

$\therefore$ $\lim_{x\to 2} f(x) = f(2)$

Hence f(x) is continuous of x = 2.

(ii) At $x = -2$

(a) $f(-2) = 3(-2) = -6$ (Defined)

(b) $\text{L.H.L.} = \lim_{x\to -2^-} 3x = 3(-2) = -6$

$\text{R.H.L} = \lim_{x\to -2^+} (x^2 - 1) = (-2)^2 - 1 = 3$

$\therefore$ L.H.L $\neq$ R.H.L.

Hence f(x) is discontinuous at x = –2. There is no need to investigate (c).

(b) If $y = e^x \sin x$, show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$ **(5)**

Ans

$$y = e^x \sin x \qquad \text{(i)}$$

$$\frac{dy}{dx} = \frac{d}{dx}(e^x \sin x)$$

$$= e^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(e^x)$$

$$= e^x \cos x + e^x \sin x \qquad \text{(ii)}$$

$$= e^x (\sin x + \cos x)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[e^x (\sin x + \cos x)]$$

$$= e^x \frac{d}{dx}(\sin x + \cos x) + (\sin x + \cos x)\frac{d}{dx}e^x$$

$$= e^x (\cos x - \sin x) + e^x (\sin x + \cos x)$$

$$= e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x \qquad \text{(iii)}$$

$$= 2e^x \cos x$$

From (ii),

$$\frac{dy}{dx} = e^x \cos x + e^x \sin x$$

$$= e^x \cos x + y \qquad \text{(From (i))}$$

$$e^x \cos x = \frac{dy}{dx} - y$$

Putting (iv) in (iii),

$$\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx} - y\right)$$

$$= 2\frac{dy}{dx} - 2y$$

$$\frac{d^2y}{dx^2} - 2\frac{2dy}{dx} + 2y = 0 \qquad \text{Proved.}$$

Q.6.(a) Integrate $\int \frac{12}{x^3 + 8}\,dx$. **(5)**

Ans $\frac{12}{x^3 + 8} = \frac{12}{(x)^3 + (2)^3} = \frac{12}{(x + 2)(x^2 - 2x + 4)}$

Making partial fractions,

$$\frac{12}{(x + 2)(x^2 - 2x + 4)} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 - 2x + 4} \qquad \text{(i)}$$

$$12 = A(x^2 - 2x + 4) + (Bx + C)(x + 2) \quad \text{(ii)}$$

For A, let $x + 2 = 0$

$$x = -2$$

Putting $x = -2$ in eq (ii),

$$12 = A((-2)^2 - 2(-2) + 4) + 0$$

$$12 = A(4 + 4 + 4) + 0$$

$$12 = 12\,A$$

$$\frac{12}{12} = A$$

$$\boxed{1 = A}$$

Expanding eq. (ii),

$$12 = Ax^2 - 2Ax + 4A + Bx^2 + 2Bx + Cx + 2C$$

$$= (A + B)x^2 + (-2A + 2B + C)\,x + (4A + 2C)$$

Comparing coefficients on both sides,

$A + B = 0$	$-2A + 2B + C = 0$	$4A + 2C = 12$
$1 + B = 0$	$-2(1) + 2(-1) + C = 0$	
$\boxed{B = -1}$	$\boxed{C = 4}$	

Putting the values of A, B and C in (i),

$$\frac{12}{(x + 2)(x^2 - 2x + 4)} = \frac{1}{x + 2} + \frac{-x + 4}{x^2 - 2x + 4}$$

$$= \frac{1}{x+2} - \frac{x-4}{x^2-2x+4}$$

$$\int \frac{12}{(x+2)(x^2-2x+4)} = \int \frac{d}{x+2} - \int \frac{x-1-3}{x^2-2x+4}\,dx$$

$$= \ln|x+2| - \frac{1}{2}\int \frac{2x-2}{x^2-2x+4}\,dx + 3\int \frac{dx}{x^2-2x+4}$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + 3\int \frac{dx}{x^2-2\,.\,x\,.\,1+(1)^2+3}$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + 3\int \frac{dx}{(x-1)^2+(\sqrt{3})^2}$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + 3\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right)\right] + C$$

$$= \ln|x+2| - \frac{1}{2}|x^2-2x+4| + \sqrt{3}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + C$$

(b) Find equations of two parallel lines, perpendicular to $2x - y + 3 = 0$ such that the product of the x- and y-intercepts of each is 3. (5)

Ans $2x - y + 3 = 0$

$$-y = -2x - 3$$
$$y = 2x + 3$$

$\therefore$ Slope of required lines $= m = -\frac{1}{2}$

Equation of required lines are:

$$y = mx + c$$
$$= -\frac{1}{2}x + c = \frac{-x+2c}{2}$$
$$2y = -x + 2c$$

or $x + 2y - 2c = 0$ (i)

For x-intercept, put $y = 0$ in (i),

$\therefore$ $x + 0 - 2c = 0$

$$x = 2c$$

For y-intercept, put $x = 0$ in (i),

$$2y - 2c = 0$$
$$2y = 2c \Rightarrow y = c$$

As product of x and y-intercept = 3

$$(2c)(c) = 3$$
$$2c^2 = 3$$

$$c^2 = \frac{3}{2}$$

$$c = \pm\sqrt{\frac{3}{2}}$$

Putting it in (1),

$$x + 2y - 2\left(\pm\sqrt{\frac{3}{2}}\right) = 0$$

$$\Rightarrow \quad x + 2y \pm \sqrt{6} = 0$$

$$\therefore \quad x + 2y + \sqrt{6} = 0 \quad \text{and} \quad x + 2y - \sqrt{6} = 0$$

Q.7.(a) Evaluate the definite integral $\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x(2 + \sin x)} dx$. (5)

Ans Let $\sin x = t \Rightarrow \cos x\, dx = dt$

Limits: Let $x = \frac{\pi}{6} \Rightarrow t = \sin\frac{\pi}{6} = \frac{1}{2}$

Let $x = \frac{\pi}{2} \Rightarrow t = \sin\frac{\pi}{2} = 1$

$$\int_{\pi/6}^{\pi/2} \frac{\cos x\, dx}{\sin x(2 + \sin x)} = \int_{1/2}^{1} \frac{dt}{t(2 + t)} \qquad \text{(i)}$$

Making partial fractions,

$$\frac{1}{t(2+t)} = \frac{A}{t} + \frac{B}{2+t} \qquad \text{(ii)}$$

$$1 = A(2 + t) + B(t) \qquad \text{(iii)}$$

For A, let $t = 0$

$$1 = A(2 + 0)$$

$$2A = 1$$

$$\boxed{A = \frac{1}{2}}$$

For B, let $2 + t = 0$

$$t = -2$$

Putting the value of t in eq. (iii),

$$1 = 0 - 2B \Rightarrow B = -\frac{1}{2}$$

Putting the value of A and B in eq. (ii),

$$\frac{1}{t(2+t)} = \frac{1}{2t} - \frac{1}{2(2+t)}$$

Putting it in eq. (i),

$$\int_{\pi/6}^{\pi/2} \frac{\cos x\, dx}{\sin x(2+\sin x)} = \int_{1/2}^{1} \frac{dt}{t(2+t)}$$

$$= \frac{1}{2}\int_{1/2}^{1} \frac{dt}{t} - \frac{1}{2}\int_{1/2}^{1} \frac{dt}{2+t}$$

$$= \frac{1}{2}[\ln|t|]_{1/2}^{1} - \frac{1}{2}[\ln|2+t|]_{1/2}^{1}$$

$$= \frac{1}{2}\left[\ln(1) - \ln\left(\frac{1}{2}\right)\right] - \frac{1}{2}\left[\ln(2+1) - \ln\left(2+\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}[0 - \ln(2)^{-1}] - \frac{1}{2}\left[\ln(3) - \ln\left(\frac{5}{2}\right)\right]$$

$$= \frac{1}{2}[\ln(2)] - \frac{1}{2}[\ln(3) - \ln(5) + \ln(2)]$$

$$= \frac{1}{2}[\ln(2) - \ln(3) + \ln(5) - \ln(2)]$$

$$= \frac{1}{2}\ln\left(\frac{5}{3}\right)$$

(b) Minimize $z = 2x + y$ subject to the constraints (5)

$x + y \geq 3$, $7x + 5y \leq 35$, $x \geq 0$, $y \geq 0$

Ans

$x + y \geq 3$ (i)
$x + y = 3$ (iii)
Putting $x = 0$ in (iii),
$0 + y = 3 \Rightarrow y = 3$
$\therefore$ (0, 3) is a point on (iii).
Putting $y = 0$ in (iii),
$x + 0 = 3 \Rightarrow x = 3$
$\therefore$ (3, 0) is another point on (iii).
Putting $x = 0, y = 0$ in (i),
$0 + 0 > 3$
$0 > 3$
Which is false. Hence solution region of (i) does not lie on the origin-side of (i).

$7x + 5y \leq 35$ (ii)
$7x + 5y = 35$ (iv)
Putting $y = 0$ in (iv),
$0 + 5y = 35 \Rightarrow y = 7$
$\therefore$ (0, 7) is a point on (iv).
Putting $x = 0$ in (iv),
$7x + 0 = 35 \Rightarrow x = 5$
$\therefore$ (5, 0) is another point on (iv).
Putting $x = 0, y = 0$ in (ii),
$0 + 0 < 35$
$\therefore$ $0 < 35$
Which is true. Hence solution region of (ii) lies on the origin-side of (ii).

Q.8.(a) Find equation of the line through the point (2, –9) and intersection of the lines (5)

$2x + 5y - 8 = 0$
$3x - 4y - 6 = 0$

Ans

$$2x + 5y - 8 = 0 \quad \text{(i)}$$

$$3x - 4y - 6 = 0 \quad \text{(ii)}$$

From (i),

$$2x = 8 - 5y$$

$$x = \frac{8 - 5y}{2} \quad \text{(iii)}$$

Putting eq. (iii) to eq. (ii),

$$3\left(\frac{8 - 5y}{2}\right) - 4y = 6$$

$$\frac{24 - 15y - 8y}{2} = 6$$

$$-23y + 24 = 12$$

$$-23y = -12$$

$$\therefore \quad y = \frac{12}{23}$$

Putting the value of y in eq. (iii),

$$x = \frac{8 - 5\left(\frac{12}{23}\right)}{2} = \frac{\frac{184 - 60}{23}}{2}$$

$$= \frac{124}{46} = \frac{62}{23}$$

$\therefore \quad \left(\frac{62}{23}, \frac{12}{23}\right)$ is the point of intersection of (i) and (ii).

Equation of line through (2, −9) and $\left(\frac{62}{23}, \frac{12}{23}\right)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y + 9 = \frac{\frac{12}{23} + 9}{\frac{62}{23} - 2}(x - 2)$$

$$= \frac{\frac{12 + 207}{23}}{\frac{62 - 46}{23}}(x - 2)$$

$$y + 9 = \frac{219}{16}(x - 2)$$

$$16y + 144 = 219x - 438$$

$$219x - 16y - 582 = 0$$

(b) **Show that the circles $x^2 + y^2 + 2x - 2y - 7 = 0$ and $x^2 + y^2 - 6x + 4y + 9 = 0$ touch externally.** **(5)**

Ans $x^2 + y^2 + 2x - 2y - 7 = 0$

$x^2 + y^2 + 2(1)x + 2(-1)y + (-7) = 0$

$g_1 = 1,\ f_1 = -1,\ c_1 = -7$

$$c_1'(-g_1, -f_1) = c_1'(-1, 1)$$

$$r_1 = \sqrt{g_1^2 + f_1^2 - c_1}$$

$$= \sqrt{(1)^2 + (-1)^2 - (-7)}$$

$$= \sqrt{1 + 1 + 7} = \sqrt{9} = 3$$

$x^2 + y^2 - 6x + 4y + 9 = 0$

$x^2 + y^2 + 2(-3)x + 2(2)y + 9 = 0$

$g_2 = -3,\ f_2 = (2),\ c_2 = 9$

$$c_2'(-g_2, -f_2) = c_2'(3, -2)$$

$$r_2 = \sqrt{g_2^2 + f_2^2 - c_2}$$

$$= \sqrt{(-3)^2 + (2)^2 - 9}$$

$$= \sqrt{9 + 4 - 9} = \sqrt{4} = 2$$

$$|c_1'c_2'| = r_1 + r_2 = 3 + 2 = 5 \qquad \text{(i)}$$

Also, $|c_1'c_2'| = \sqrt{(3 + 1)^2 + (-2 - 1)^2}$

$$= \sqrt{16 + 9} = \sqrt{25} = 5 \qquad \text{(ii)}$$

From (i) and (ii), it is prove that the given two circles touch externally.

Q.9.(a) Find an equation of the ellipse having foci $(\pm 5, 0)$ and passing through the point $\left(\frac{2}{3}, \sqrt{3}\right)$. **(5)**

Ans $F(\sqrt{5}, 0),\ F'(-\sqrt{5}, 0)$ through $\left(\frac{3}{2}, \sqrt{3}\right)$

$$a_e = \sqrt{5} \quad \Rightarrow \quad e = \frac{\sqrt{5}}{a}$$

$$e^2 = \frac{a^2 - b^2}{a^2} \quad \Rightarrow \quad \frac{5}{a^2}\ \frac{a^2 - b^2}{a^2}$$

$$= a^2 - b^2 = 5 \quad \Rightarrow \quad a^2 = 5 + b^2 \qquad \text{(i)}$$

Equation of required ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{(ii)}$$

Since $\left(\frac{3}{2}, \sqrt{3}\right)$ lie on it, therefore

$$\frac{\frac{9}{4}}{a^2} + \frac{3}{b^2} = 1 \quad \Rightarrow \quad \frac{9}{4a^2} + \frac{3}{b^2} = 1 \quad \Rightarrow \quad \frac{9b^2 + 12a^2}{4a^2b^2} = 1$$

$$\therefore \quad 12a^2 + 9b^2 = 4a^2b^2 \qquad \text{(iii)}$$

Putting the value of eq. (i) in eq. (iii),

$$12(5 + b^2) + 9b^2 = 4b^2(5 + b^2)$$

$$60 + 12b^2 + 9b^2 = 20b^2 + 4b^4$$
$$0 = 4b^4 - b^2 - 60$$
Let $b^2 = t$
$$4t^2 - t - 60 = 0$$
$$t = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-60)}}{2 \times 4} = \frac{1 \pm \sqrt{1 + 960}}{8}$$
$$= \frac{1 \pm 31}{8} \Rightarrow \frac{1 + 31}{8} \quad \& \quad t = \frac{1 - 31}{8}$$
$$= \frac{32}{8} \qquad = -\frac{30}{8}$$
$$t = 4, -\frac{15}{4} \qquad \therefore \quad -\frac{15}{4}$$
Gives imaginary roots, hence discard it.

$\therefore \quad b^2 = t = 4 \Rightarrow b = \pm 2$

Putting the values in (i),
$$a^2 = 5 + 4 = 9 \Rightarrow a = \pm 3$$
Putting the values in (ii),
$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

(b) A particle acted upon by constant forces $4\underline{i} + \underline{j} - 3\underline{k}$ and $3\underline{i} - \underline{j} - \underline{k}$ is displaced from A(1, 2, 3) to B(5, 4, 1). Find the work done. (5)

Ans Let $\vec{F_1} = 4\underline{i} + \underline{j} - 3\underline{k}$ and $\vec{F_2} = 3\underline{i} - \underline{j} - \underline{k}$

Total force $= \vec{F} = \vec{F_1} + \vec{F_2}$
$$= (4\underline{i} + \underline{j} - 3\underline{k}) + (3\underline{i} - \underline{j} - \underline{k})$$
$$= 7\underline{i} + 0\underline{j} - 4\underline{k}$$
$$\vec{d} = \vec{AB} = (5 - 1)\,\underline{i} + (4 - 2)\,\underline{j} + (1 - 3)\,\underline{k}$$
$$= 4\underline{i} + 2\underline{j} - 2\underline{k}$$
Work done $= \vec{F}.\vec{d} = (7\underline{i} + 0\underline{j} - 4\underline{k})(4\underline{i} + 2\underline{j} - 2\underline{k})$
$$= (7 \times 4) + (0 \times 2) + (-4 \times -2)$$
$$= 28 + 0 + 8 = 36$$